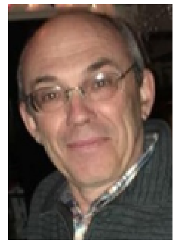
What Inventive Principles were Used by Euclid in the Proof of the Proposition 1-47 Otherwise Known as Pythagorean Theorem

by Igor Polkovnikov, 2019



Abstract. In the proof of the famous Proposition 1-47, Euclid used several well known TRIZ inventive principles as well as clearly establishing the Ideal Final Result and introducing an X-element. In this example, it is discussed what systems and their hierarchies, from an inventive point of view could be in mathematics, what is an interaction and what is a field (in TRIZ terms). The line of proof wiggling from a system level to a system level is shown
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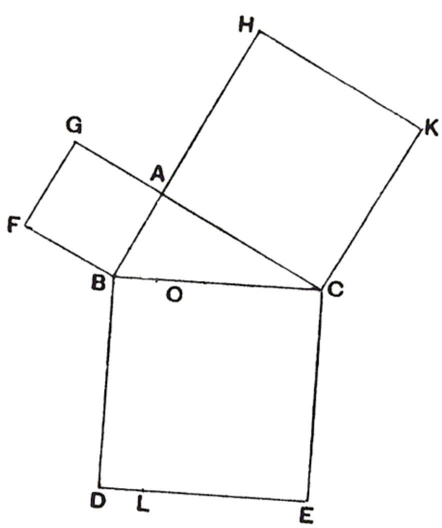
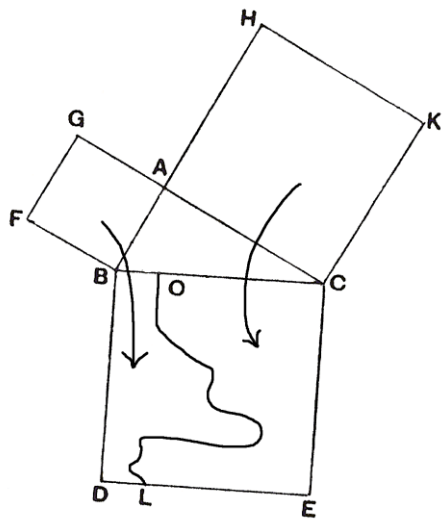
The Purpose of this paper is to find common grounds for mathematics and system science with the great discoveries made in TRIZ. These common grounds are required 1). to introduce TRIZ systematic methods into other disciplines and 2). to be able to investigate such disciplines for the TRIZ purpose, i.e. making new inventions in the disciplines themselves and applying the deep mathematical knowledge into practical life of an inventor beyond what is usual today.

Figure 1



Situation. (Figure 1) Mathematical proof involves a process of "construction", otherwise, invention. Let us see what inventive principles Euclid used in his proof of the Proposition 47 from Book 1. English translation reads the theorem statement like this: "In right-angled triangles the square on the side subtending the right angle is equal to the squares on the sides containing the right angle." It means, the area of the squares constructed on the catheti of the right-angled triangle is equal the area of the square constructed on the hypotenuse. On the first figure, the squares ABFG and AHKC are the catheti squares of the right-angled triangle ACB and BCED is a hypotenuse triangle. That is what is given as a problem situation.

Figure 2



Ideal Final Result (IFR), or, in other words, plan of the solution, is if BCED is divided, or consist of two figures BOLD and OCEL (not necessary squares) which respectively are equal to ABFG and AHKC, then it proves the theorem (Figure 2). Contradiction here is "catheti squares must match or be located over two figures which BCEL consists of, but they are located elsewhere". What is supposed to be devised is the mechanism of placing catheti square areas onto the parts of the hypotenuse square. Euclid proposes the "X-Element" which is the cornerstone of the solution right away, but I am going to postpone its introduction.

Clarification. A good chunk (about 25%) of Euclid's proof is dedicated to proving that AG is on the same line as AC (and correspondingly AB and AH). I am omitting it here. This is the clarification of situation in order to find all parts, their interactions, and governing rules. Clarification can be done as an initial step in problem consideration as well as after the IFR and the general idea of the solution are selected. In practice, Clarification may take 10 to 100 times more time and effort than the actual solution process. While "troubleshooting" of industrial equipment, once a problem is found, it is practically solved. In medicine, it is called Diagnosis.

System Interaction in Mathematics. Before I tell you about the initial step of the solution, let us speculate about the nature of Interaction in Mathematics. In other disciplines, interaction involves time. But there is no notion of time in Math. If we want to spread the system approach to Math, we have to find, or define, how Math systems interact. When systems interact, there is always something common amongst them. It could be a boundary or a "channel" which is a form of boundary. One can say that the boundary belongs to both interacting systems, or forms bi-systems with them.

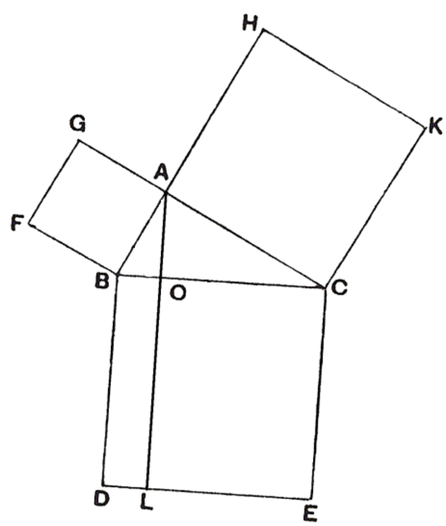
With this, through the point B, the square ABFG interacts with BCED, and through the point C, AHKC interacts with the same BCED. An interaction point should play a role since it belongs to both systems. We can attempt to build our proof mechanism around it. (In accordance with my book "Junctions", this could be called a Junction).

Let us look on a slightly larger scale. On one hand we have a bi-system of catheti squares ABFG and AHKC which interact through point A. On another hand we have a hypotenuse bi-system consisting of BOLD and OCEL which interact through a line OL.

We may suppose that in our future mechanism of matching, or translation, the point A should match the line OL. On an even larger scale, the catheti squares bi-system interacts with BCED through the triangle ACB. The triangle ACB is a boundary between the squares and should "pass interactions" occurring among them.

Initial Step. What should be introduced to start construction, what is the best initial step? It is always a choice. The first step is usually faulty. But it allows investigation and makes the next steps more targeted. No doubt, Euclid had these dilemmas. Let us see what qualities possess the choice he finally proposed. What OL should be? Obviously, a seemingly simplest one, a straight line. But a straight line is not as simple as what we are accustomed to think. It carries with it, through its possible interactions with other objects, a wide range of useful properties abundantly described in the previous propositions of the Euclid book. A straight line is a rich element. In dialectical terms, it is both simple and complex. There are infinitely many straight lines possibly intersecting BCED and thus dividing it. Narrowing the search, we want to put it in relation to catheti squares, presumably both. There is one convenient point: the point A. An AL straight line will divide the BCED and the catheti bi-system. (Likewise, it divides the triangle ACB, but that is not used further). How should AL divide BCED? There are only three ways a line can divide a square: 1). through vertices dividing it into two equal triangles (this can not be used if the point A is chosen); 2). dividing the square under an angle; and 3). be parallel to the sides, dividing the square into parallelograms. If we choose 3), AL becomes parallel to BD and CE. Parallel lines, when crossed with other lines (and we have it), carry abundance of area-related properties and effects. Let us do it. This is the cornerstone of our construction.

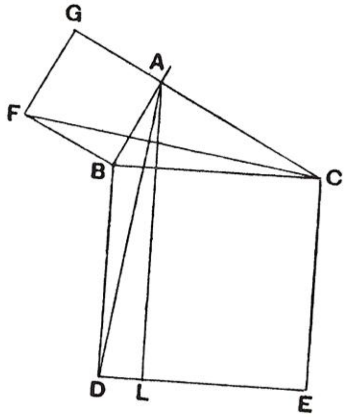
Figure 3



It is worth noting that the line AL, in this case, unifies as its parts the point A and the segment OL which as we mentioned earlier correspond to each other.

The straight line AL is parallel to BD and CE and divides the line BC in the point O (Figure 3). Another interesting fact (but not used further): the construction has two pairs of lines intersecting at the right angles: BH and CG intersecting at A, and AL and BC intersecting at O.

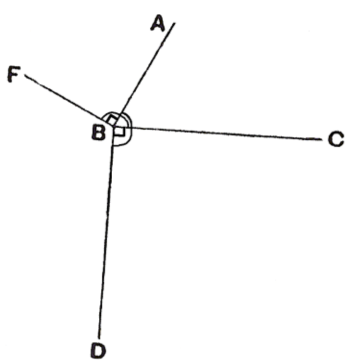
Figure 4



Symmetry-Asymmetry. The initial construction is size-wise asymmetrical, though structurally symmetrical. We may remove the right part, work only with the left side, and prove ABFG=BOLD which is sufficient (Figure 4). This is one more achievement obtained with the parallel AL. The whole BCED should stay since it, as the whole, defines the properties of its parts used further.

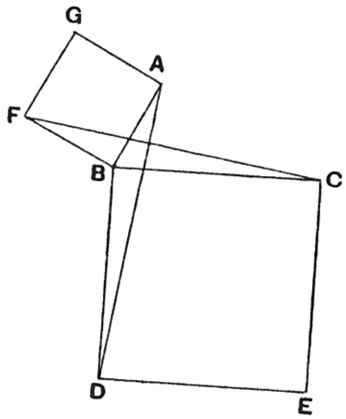
Construction of the Proof. In the basic geometry, besides points, lines, and figures with areas, there are also angles. Their main property is that they can be added and subtracted and the larger is greater than its parts. Like line segments related to infinite lines and dividing them infinitely, angles are related to the opposite of the line: a circle. There are an infinite number of segments on a line with not-connected ends, and a finite number of angles in a circle (a line with connected ends, "endless" in totally different way than a straight line), precisely equal to four right angles. In relation to angles, it does not matter what the size of a circle is. A system made of angles is the next "part" introduced to the construction.

Figure 5



Interaction Through Angles. Looking at angles (Figure 5), there are three of specific interest: FBA, an angle of a cathetus, CBD, an angle of the hypotenuse, and ABC, an angle of the triangle between the former two. The square BCED interacts with the square ABFG not only through the point B, but also through the angle ABC. The angle FBA as the part of ABFG forms a bi-system with ABC obtaining the system of angles FBC. Likewise, the angle CBD as the part of BCED forms a bi-system with ABC obtaining the system of angles ABD. These angles, FBC and ABD are new systems, one part of each belonging to the original objects, and another, namely, ABC is common. This is a more complex method of interactions of original objects. And concluding from the mathematical common notion that "if equals are added to equals, the wholes are equal", we derive a property of this interaction. In the more simple and familiar mathematical language: "Since ABF=CBD: for each is right: let the angle ABC be added to each; therefore the whole FBC=ABD". In the Altshuller's Principles, equality likewise plays an important part. The Principle 1 deals with objects by dividing into or unification of equal parts. Equal defines unequal.

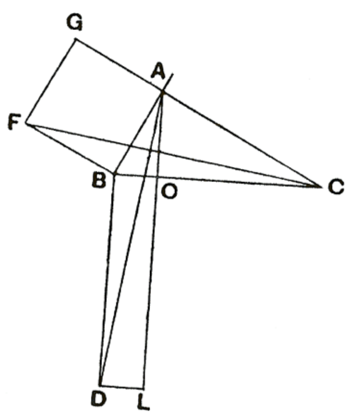
Figure 6



From Angles to Triangles. The next elements in the construction are triangles BCF and BAD (Figure 6). Contrary to points, lines, and angels, they have area, a target of our interest. These triangles consist of parts belonging to both squares. Each triangle has a short side belonging to the smaller cathetus square, and long side belonging to the larger square of hypotenuse. They have a common vertice B which belongs to every figure. As it was figured above, the angles FBC=ABD. That, by the Euclid theorem 1.4, makes the triangles congruent. Now another required property appears: equality of areas.

The Same, but Different. Triangles BCF and BAD are different, but from another point of view, it is the same triangle rotated around B. In the static consideration the triangles are different, if allowing mathematical transformations - they are the same, the same thing in two incarnations. In the terms of the "Junctions" book, these are two different system worlds.

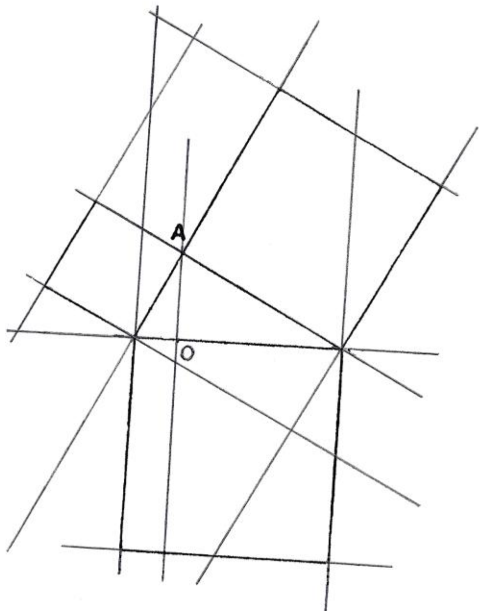
Figure 7



Yet Another View on the System: The Final Push (Figure 7). Figures ABFG and BCF are contained between two parallel lines CG and BF, and have the same base: BF. For such figures there is a property: the area of CBF=1/2 of the area of ABFG (or, in a more common form, the area of BCF=the area of ABF, and ABFG=2ABF). And the same for the parallel lines AL and BD: ADB=1/2BOLD.

Therefore, BOLD=ABFG. Making the same process with the right part, concluding, BCED=ABFG+ACKH. The proof is finished.   Q.E.D.

Figure 8



Parallel lines were already a canvas on which our triangle and squares were incorporated (Figure 8). We have made use of that laying the final block into the building.

System Levels Line of the Proof. In geometry, the system hierarchy can be built this way: 1) a point; 2) two points form a system with the system property: distance; 3) infinite number of points + some rules form a line, a straight one in our case; 4) two straight lines and a point form an angle (note that angles are measured by distances of three pairs of points in a triangle); 5) a triangle as a collection of lower systems consisting of lines, points, three distances and three angles; 6) a triangle as a plane figure with area, it is a higher systematic level. Area is a system property of plane figures. 7) a square as a collection of points, lines, angles, and distances ("sides are equal"). 8) A square having a system property "area". There is another system used: a pair of parallel lines. I have difficulty defining its hierarchical level, we can assume that it is an the level of angles, 4).

The line of proof is this: 8) squares with areas --> 7) non-area properties of a square --> 4) angles --> 5) triangles (not working with area) --> 4) parallel lines --> 8) triangles and squares as plain figures with area. I see it as jumping from a systematic level down to a lower level (done three times) and up. In other words, the proof follows the general idea of switching to lower levels, finding a conclusion there and returning up.

Two pairs of parallel lines used in the proof should be a higher level than 7). Their intersection is a parallelepiped, which a square is. The pair of parallel lines is on a higher level than the square, since the square is the system property of the pair. It is difficult to set up what is on the higher level in mathematics: a pair of points define a line, but a line consist of points. Both are valid statements. And it can be used in both ways. We do use two pairs of parallel lines here, but only in relation to areas of a containing square and a triangle. It can be viewed in another way: when a square and a triangle are put within the canvas of certain parallel lines, their areas follow the strict rule ABFG=2ABF.

There is one more mechanism of interactions. There is no time in mathematics, but the process of our thought take time. There is a sequence of logical steps in proofs and constructions. The order of those are the "timelines" which can be a basis of interaction and establishment of a system hierarchy (choosing what is a part of what) in mathematics.

Literature

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Author Biography. Igor Polkovnikov has been working as a software engineer, automatic system engineer, and image analysis expert since 1982. Some companies he worked for: Paramount Pictures, Incyte Genomics, A&B Software, Intuitive Surgical, Infosys, KLA, Brooks Automation, and Yes San Francisco. Igor is an inventor of what is now called "Google street view". The first virtual tour was published in 2002. He is the author of three books and several articles. Igor is an accomplished Argentine Tango dancer and has been teaching at his dance school in Palo Alto for 14 years. Igor attended TRIZ seminars in Minsk in 1980-1984. Since 2018 he is an organizer of a bi-weekly "Problem Solving Innovators Club" in Palo Alto, California. Igor's scientific interests are: kinesiology applied to dance, intersection of systems science with TRIZ and mathematics, history, memorization hints for guitarists, automatic diagramming of software systems, asynchronous neural networks for image analysis, theory of virtual tours.